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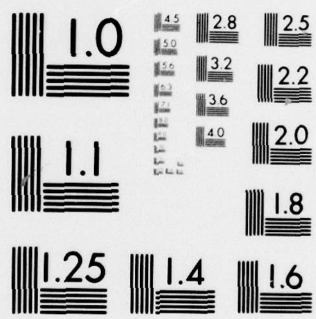

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A STOCHASTIC REPRESENTATION FOR THE PRINCIPAL EIGENVALUE OF A SECOND-ORDER DIFFERENTIAL EQUATION+

by

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November 1979

Submitted to "Stochastics"

+This research was supported by the Air Force Office of Scientific Research under AF [redacted]
AFOSR-76-3063

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

19 REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 18 AFOSR-TR-80-0048

2. GOVT ACCESSION NO. Principal

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle): 6 A STOCHASTIC REPRESENTATION FOR THE ~~PRINCIPLE~~ EIGENVALUE OF A SECOND-ORDER DIFFERENTIAL EQUATION.

5. TYPE OF REPORT & PERIOD COVERED: 9 INTERIM rept.

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s): 10 Ioannis/Karatzas

8. CONTRACT OR GRANT NUMBER(s): 15 AFOSR-76-3063

9. PERFORMING ORGANIZATION NAME AND ADDRESS:
Brown University
Division of Applied Mathematics
Providence, Rhode Island 02912

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 16 61120F 2304/A1 17 A1

11. CONTROLLING OFFICE NAME AND ADDRESS:
Air Force Office of Scientific Research/NM
Bolling AFB, Washington, DC 20332

12. REPORT DATE: 11 Nov[redacted]979

13. NUMBER OF PAGES: 24

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): 12 25

15. SECURITY CLASS. (of this report): UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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Unclassified

A STOCHASTIC REPRESENTATION FOR THE PRINCIPAL EIGENVALUE OF A SECOND-ORDER DIFFERENTIAL EQUATION+

Ioannis Karatzas

Abstract

Using ideas from stochastic control we derive a stochastic representation for the smallest eigenvalue of a second-order differential equation. As a side-result we solve an associated stationary control problem in a very general setting.

1. INTRODUCTION AND SUMMARY

In the present paper we obtain a stochastic representation for the principal eigenvalue λ^* of the eigenvalue problem

$$z''(x) + 2(\lambda-\phi(x))z(x) = 0, \quad \text{in } \mathbb{R} \tag{1}$$

with $z(\cdot)$ even and $\phi(\cdot)$ a positive, even, C^1 and convex function on the reals, growing like x^{2m}, $m \geq 1$ to infinity as $|x| \to \infty$. Under these assumptions on $\phi(\cdot)$ there exists a discrete spectrum for problem (1) (see Titchmarsh [15], vol. 1 and Coddington and Levinson [2]), with eigenvalues $0 < \lambda^* = \lambda_0 < \lambda_1 < \ldots < \lambda_n < \ldots$, $\lambda_n \to \infty$ as $n \to \infty$ and corresponding eigenfunctions $z_n(x) \in L^2(0,\infty)$, $n \in \mathbb{N}$, having exactly n zeros on $(0,\infty)$. We prove that λ^* admits the representation

$$\lambda^* = - \lim_{\tau\to\infty} \frac{1}{\tau} \ln E\left[\exp\{-\int_0^\tau \phi(x+w_s)ds\}\right] \tag{2}$$

for any $x \in \mathbb{R}$, where $\{w_t;\ t \geq 0\}$ is a Brownian Motion process on an underlying probability space $(\Omega, \mathcal{F}, P)$ and E denotes expectation with respect to P.

The method proceeds by considering a stationary stochastic control problem associated in some natural way with equation (1), along with the corresponding family of finite-horizon optimization problems. It is proven that λ^* is the optimal asymptotic performance rate for the statbnary control problem. A major step is then to show that $V(x,\tau)$, the optimal expected performance for the control problem on the finite time interval $[0,\tau]$, satisfies the limiting relationship

$$\lim_{\tau\to\infty} \frac{V(x,\tau)}{\tau} = \lambda^*, \text{ any } x \in \mathbb{R}$$

(Theorem 1). Equation (2) follows from a Feynman-Kac-type stochastic representation for $V(x,\tau)$. On the other hand, a candidate for the optimal law in the stationary problem is discerned from the corresponding Bellman equation and its optimality against any admissible nonanticipative law is established by means of the aforementioned limiting relationship.

Donsker and Varadhan [3] and Holland [8,9] obtained different representations for the smallest eigenvalue of second-order elliptinc differential operators in bounded domains under natural or Dirichlet boundary conditions. Holland [9], in particular, established a generalized Rayleigh-Ritz formula in this setting, using ideas from stationary stochastic control. His method did not, however, invoke the corresponding finite-horizon problems. Stationary control problems were studied in some generality by Wonham [16] and Kushner [11]. Both restricted attention to laws producing an ergodic diffusion process. For an example of a stationary control problem in which general nonanticipative laws are admitted, see Beneš and Karatzas [1].

2. THE STATIONARY CONTROL PROBLEM

Denote by $z^*(\cdot)$ the corresponding to λ^* eigenfunction of equation (1), normalized so that $\int_{-\infty}^{\infty} (z^*(x))^2 dx = 1$. The change of variables $z^* = \exp(-v)$, $v(\cdot)$ an even, positive function on $\mathbb{R}$, yields the equation

$$\lambda^* = \frac{1}{2} v_{xx}(x) - \frac{1}{2} v_x^2(x) + \phi(x), \text{ in } \mathbb{R}. \tag{3}$$

Note that $-\frac{1}{2} v_x^2 = \min_u (u \cdot v_x + \frac{1}{2} u^2)$, the minimum being achieved by $u^* = -v_x$. It becomes transparent that (3) is the Bellman equation for a stationary control problem - see Wonham [16] - to be introduced in detail below.

Consider the space $C[0,T]$ of real-valued, continuous functions on $[0,T]$, some $T > 0$, and let Σ_t; $0 \leq t \leq T$ denote the -field of subsets of $C[0,T]$ generated by $\{x; x \in C[0,T], s \leq t\}$. Consider also the σ-field $\mathscr{D}$ of subsets D of $[0,T] \times C[0,T]$ with the property that, for any $0 \leq t \leq T$, the section D_t belongs to Σ_t, and that each x-section D_x, $x \in C[0,T]$ is Lebesgue-measurable.

<u>Definition 1</u>: An admissible nonanticipative control law u is a measurable function

$$u: ([0,T] \times C[0,T], \mathscr{D}) \to (\mathbb{R}, \mathbb{B})$$

such that the stochastic differential equation

$$\begin{aligned} dx_t &= u_t(x)dt + dw_t \\ x_0 &= x \end{aligned} \tag{4}$$

have a unique (in the sense of the probability law) weak solution $\{(x_t, w_t);\ 0 \le t \le T\}$ on some probability space $(\Omega, \mathcal{F}_T, P_x^u; \mathcal{F}_t)$, for any $x \in \mathbb{R}$, with $\{w_t;\ t \ge 0\}$ an $\mathcal{F}_t$-adapted Brownian Motion process and

$$E_x^u \int_0^T |u_t(x)|^p dt < \infty \tag{5}$$

$$\sup_{0 \le t \le T} E|x_t|^p < \infty \tag{6}$$

holding for any $p > 0$. Let $\mathcal{U}$ denote the class of all admissible nonanticipative control laws.

For instance, if u is bounded such a solution can be constructed via the Girsanov substitution of measures (see Liptser and Shiryayev [12]). If u is a function of (t, x_t) and satisfies a local Lipschitz and a global linear growth condition in the space variable then a solution of (4) exists in the strong Itô sense.

The optimal stationary control problem can now be formulated as follows: choose a law $u^* \in \mathcal{U}$ for which the limit

$$J(u^*, x) = \lim_{T\to\infty} \frac{1}{T} E_x^* \int_0^T \{\phi(x_t^*) + \frac{1}{2} (u_t^*)^2\} dt$$

exists and does not exceed the average expected total cost

$$J(u,x) = \varliminf_{T\to\infty} \frac{1}{T} E_x^u \int_0^T \{\phi(x_t^u) + \frac{1}{2} (u_t(x^u))^2\} dt \tag{7}$$

of starting at place x and exerting control law u for all $x \in \mathbb{R}$, $u \in \mathcal{U}$.

A natural subclass of $\mathscr{U}$ for this problem consists of the markovian laws that give rise to an ergodic solution process (x_t) in (4). Those laws are of the form $u_t(x) = b(x_t)$, $b: \mathbb{R} \to \mathbb{R}$ measurable and such that $F^u(\infty) < \infty$, with

$$F^u(x) \triangleq \int_{-\infty}^{x} \exp\{2 \int_0^y b(z)dz\}dy. \tag{8}$$

It is proved in Gihman and Skorohod [7; §18] that the process (x_t^u), corresponding to such a law u, exists in the strong sense and admits $F^u(x)/F^u(\infty)$ as a (unique) invariant probability distribution function, in the sense that

$$\int_{-\infty}^{\infty} P(x_t^u \leq y | x_0^u = x)dF^u(x) = F^u(y); \quad 0 \leq t \leq T, \; y \in \mathbb{R} \tag{9}$$

$$\frac{1}{T}\int_0^T f(x_t^u)dt \xrightarrow[T\to\infty]{} \frac{1}{F^u(\infty)} \int_{-\infty}^{\infty} f(y)\; dF^u(y); \text{ a.s. (P) and } L'(E), \tag{10}$$

uniformly on bounded x sets.

$$\lim_{t\to\infty} E[f(x_t^u) | x_0^u = x] = \frac{1}{F^u(\infty)} \int_{-\infty}^{\infty} f(y)dF^u(y) \tag{11}$$

for any Borel measurable function $f(\cdot)$ for which the integrals in (10), (11) exist, and any $x \in \mathbb{R}$.

<u>Definition 2</u>: Let $\mathscr{S}$ be the class of laws of the form $u_t(x) = b(x_t)$ such that both $F^u(\infty) < \infty$ and

$$\int_{-\infty}^{\infty} \{\phi(x) + b^2(x)\}dF^u(x) < \infty \tag{12}$$

are satisfied.

It is seen from (10) that the limit in the performance index

$$J(u) = \lim_{T\to\infty} \frac{1}{T} E \int_0^T \{\phi(x_t^u) + \frac{1}{2} u_t^2(x^u)\} dt \qquad (7)'$$

exists and is independent of the starting point $x \in \mathbb{R}$.

LEMMA 1: <u>For the solution $v(\cdot)$ of equation (3) with $v(0) = 0$,</u>

$$\underline{\operatorname{sgn} v_x(x) = \operatorname{sgn} x} \qquad (13)$$

$$\underline{v_x(|x|) \sim |x|^m, \text{ as } |x| \to \infty.} \qquad (14)$$

<u>Proof</u>: Consider $d > 0$ such that $\phi(d) = \lambda^*$ and note that $(z^*(x))''$ is negative on $(0,d)$, positive on (d,∞). Since $z^{*\prime}(0) = \lim_{x\to\infty} (z^*(x))' = 0$, this implies $(z^*(x))' > 0$, hence $v_x(x) > 0$, on $(0,\infty)$. (13) follows by oddness of $v_x(x)$ in x, while (14) is a consequence of equation (3) and the corresponding growth condition on $\phi(\cdot)$, q.e.d.

Consider now a probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ and an $\mathcal{F}_t$-adapted Brownian Motion process $\{w_t;\ t \geq 0\}$ defined on it. The stochastic differential equation

$$\begin{aligned} d\xi_t &= -v_x(\xi_t)dt + dw_t;\ t \geq 0 \\ \xi_0 &= x \end{aligned} \qquad (15)$$

has a strong solution on $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ because of (B); see Proposition 2 below. The implicit minimization in (3) suggests that

$\{\xi_t;\ t \geq 0\}$ achieves the infimum of $J(u)$ in (7)', $u \in \mathscr{U}$; in fact we have the following result:

PROPOSITION 1: <u>The control law $u^*_t(x) = -v_x(x_t)$ is optimal in $\mathscr{U}$.</u>

<u>Proof</u>: First it is checked that $u^* \in \mathscr{U}$. In fact,

$$F^*(x) = \int_{-\infty}^{x} \exp\{-2 \int_0^y v_x(\zeta)d\zeta\}dy = \int_{-\infty}^{x} (z^*(y))^2 dy, \quad F^*(\infty) = 1$$

and thus condition (12) follows from

$$\int_{-\infty}^{\infty} \{\phi(x)(z^*(x))^2 + \frac{1}{2}(z^{*\prime}(x))^2\}dx = \lambda^*, \tag{16}$$

which, in turn, is easily obtained from (1) with z,λ replaced by z^*,λ^* upon multiplying throughout by z^* and integrating over $\mathbb{R}$. Conditions (5) and (6) are also readily verifiable; see (24) below.

To prove optimality of u^* in $\mathscr{U}$, consider any law $u \in \mathscr{U}$ and apply Itô's rule to $v(x^u_t)$; taking equation (3) along with conditions (14), (6) into account, one gets

$$Ev(x^u_t) - v(x) + E\int_0^t \{\phi(x^u_s) + \frac{1}{2}u^2_s(x^u)\}ds \geq \lambda^* t. \tag{17}$$

But $\lim_{t\to\infty} Ev(x^u_t) = (F^u(\infty))^{-1} \int_{-\infty}^{\infty} v(y)dF^u(y) < \infty$ by virtue of (11), so one obtains $J(u) \geq \lambda^*$ upon dividing both sides of (17) by t and passing to the limit as $t \to \infty$. If $u = u^*$, (17) becomes an equality and therefore

$$J(u^*) = \lim_{T\to\infty} \frac{1}{T} E \int_0^T \{\phi(\xi_t) + \frac{1}{2} v_x^2(\xi_t)\}dt = \lambda^*. \tag{18}$$

So u^* is optimal in $\mathscr{L}$.

LEMMA 2: $P(|\xi_t| \leq |x+w_t|,\ t \geq 0) = 1.$

Proof: An easy consequence of (13), Proposition 1 and the Ikeda-Watanable comparison Theorem [10].

3. A FAMILY OF FINITE-HORIZON CONTROL PROBLEMS

Under the same assumptions and definitions as in Section 2, consider the problem of finding the optimal law $u^*(x)$; $0 \leq t \leq \tau$ in $\mathscr{L}$ that minimizes the expected total cost

$$I(x,\tau;u) = E_x^u \int_0^\tau \{\phi(x_t^u) + \frac{1}{2} u_t^2(x^u)\}dt, \quad 0 \leq \tau \leq T.$$

The Bellman equation for the value function $V(x,\tau)$ on $\mathbb{R} \times [0,T]$ associated with this problem,

$$V_\tau = \frac{1}{2} V_{xx} - \frac{1}{2} V_x^2 + \phi(x) = \frac{1}{2} V_{xx} + \min_{u\in\mathbb{R}}(uV_x + \frac{u^2}{2}) + \phi(x); \ (x,\tau) \in \mathbb{R} \times (0,T] \tag{19}$$

$$V(x,0) = 0 \qquad ; x \in \mathbb{R}, \tag{20}$$

has a unique classical solution which is $C^{2,1}$ in $\mathbb{R} \times (0,T]$, continuous on $\mathbb{R} \times [0,T]$ and satisfies a polynomial growth condition in the space variable; see Fleming [4]. It is easily checked that $V(x,\tau)$ is an (achievable) lower bound on $I(x,\tau;u)$

and that it is even and convex in x for any $\tau \in [0,T]$ (for the first claim, apply Itô's rule to $V(x_t^u, \tau-t)$, $u \in \mathscr{U}$; symmetry follows from uniqueness and the fact that $W(x,\tau) = V(-x,\tau)$ is also a solution to (19), (20); for convexity, see Fleming and Rishel [6], Ex. VI. 9). The minimization in (19) is achieved for $u = -V_x$, a fact suggesting that the infimum of $I(x,\tau;u)$ is achieved by the process $\{\eta_t^\tau;\ 0 \leq t \leq \tau\}$ defined through

$$\begin{aligned} d\eta_t^\tau &= -V_x(\eta_t^\tau, \tau-t)dt + dw_t; \quad 0 \leq t \leq \tau \\ \eta_0^\tau &= x \end{aligned} \tag{21}$$

and that the optimal control law in $\mathscr{U}$ for the problems on $[0,\tau]$ is $\tilde{u}_t(x) \triangleq -V_x(x_t, \tau-t)$; $0 \leq t \leq \tau$, $x \in C[0,T]$.

PROPOSITION 2: <u>The stochastic differential equation (21) is strongly solvable on the probability space $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$.</u>

<u>Proof</u>: The symmetry and convexity properties of $V(x,\tau)$ suggest that

$$V_x(x,\tau) \geq 0, \quad \text{on} \quad x > 0. \tag{22}$$

In what follows we adapt an argument of Fleming [5]. Consider the expanding sequence of intervals $G_n = (-n,n)$ and the corresponding sequence of functions $\beta_n \in C^\infty(\mathbb{R})$; $0 \leq \beta_n \leq 1$, $\beta_n(x) = 1$ in G_n, $\beta_n(x) = 0$ in $\tilde{G}_{n+1}$. The functions $f_n(x,\tau) = -V_x(x,\tau)\beta_n(x)$ satisfy both a Lip and a linear growth condition in x, hence the process

$\{\eta_t^{\tau,n};\ 0 \le t \le \tau\}$,

$$d\eta_t^{\tau,n} = f_n(\eta^{\tau,n},\tau - t)dt + dw_t; \qquad 0 \le t \le \tau$$

$$\eta_0^{\tau,n} = x \in G_n,$$

is well defined in the Itô sense, for all sufficiently large $n \in \mathbb{N}$. Besides, if $m < n$, then $\eta_t^{\tau,m} = \eta_t^{\tau,n}$, $0 \le t \le \tau_m$, where

$$\tau_n = \inf\{t \le \tau;\ |\eta_t^{\tau,n}| \ge n\}$$

$$= \tau, \text{ if } |\eta_t^{\tau,n}| < n, \text{ all } 0 \le t \le \tau.$$

Denote by τ_∞ the a.s. limit of the increasing sequence τ_n and define $\eta_t^\tau = \eta_t^{\tau,n}$, for n large enough and $0 \le t \le \tau_\infty$. We prove that η_t^τ is defined on the whole of $[0,\tau]$, i.e. that $\tau_\infty = \tau$, a.s.

It is easily seen from (22) that the process $q_t^{\tau,n} = (\eta_t^{\tau,n})^2 + \tau - t$, $0 \le t \le \tau$ is a nonnegative supermartingale. Therefore,

$$P(\tau_n < t) \le P(\sup_{0\le s\le t} q_t^{\tau,n} > n^2) \le \frac{x^2+\tau}{n^2} \to 0 \text{ as } n \to \infty,$$

any $0 \le t \le \tau$. Consequently, $\tau_\infty = \tau$, a.s.(P); Q.E.D.

4. ASYMPTOTIC BEHAVIOUR OF THE VALUE FUNCTION

In this section we establish the main result of the paper, Theorem 1, which gives an asymptotic relationship between the optimal expected performance $V(x,\tau)$ on $[0,\tau]$ and the optimal steady-state performance rate λ^*. This relationship is used to establish the stochastic representation (2) for λ^* (Theorem 2) and to prove the optimality in $\mathscr{U}$ of the law u^* for the stationary problem (Theorem 3).

LEMMA 3: $V(x,\tau) \leq v(x) + \lambda^*\tau$; $(x,\tau) \in \mathbb{R} \times [0,T]$, any $T > 0$.

Proof: The stationary optimal law $u_t^*(x) = -v_x(x_t)$ is suboptimal for the finite-horizon problem; we have therefore from equation (19):

$$V_\tau \leq \frac{1}{2} V_{xx} + \frac{1}{2} v_x^2 - v_x V_x + \phi(x); \; (x,\tau) \in \mathbb{R} \times (0,T].$$

Consequently, the function

$$W(x,\tau) \triangleq v(x) + \lambda^*\tau - V(x,\tau)$$

satisfies the differential inequality

$$W_\tau \geq \frac{1}{2} W_{xx} - v_x W_x; \; (x,\tau) \in \mathbb{R} \times (0,T]$$

along with the initial condition $W(x,0) = v(x)$; $x \in \mathbb{R}$.

An application of Itô's rule to $W(\xi_t,\tau-t)$ gives the stochastic differential inequality $dW(\xi_t,\tau-t) \leq W_x(\xi_t,\tau-t)dw_t$, which implies that $\{W(\xi_t,\tau-t);\; 0 \leq t \leq \tau\}$ is a supermartinglae since, due to the polynomial growth condition of W_x in x and Lemma 2, the process

$$\left\{ \int_0^t W_x(\xi_s,\tau-s)dw_s;\; 0 \leq t \leq \tau \right\}$$

is a square-integrable martingale.

Introduce the sequence of stopping times

$$T_n = \inf\{0 \leq t \leq \tau;\ |\xi_t| \leq n\}$$
$$= \tau, \text{ if } |\xi_t| < n, \text{ all } 0 \leq t \leq \tau. \tag{23}$$

An application of Itô's rule to $\xi_t^{2\ell}$, $\ell \geq 1$ an integer, gives

$$d\xi_t^{2\ell} = \ell\xi_t^{2(\ell-1)}[(2\ell-1) - 2\xi_t v_x(\xi_t)]dt + 2\ell\xi_t^{2\ell-1}dw_t.$$

It is easy to check that for each $\ell \geq 1$ there exists $a_\ell > 0$ such that $\ell x^{2(\ell-1)}[(2\ell-1) - 2xv_x(x)] \leq a_\ell$, $x \in \mathbb{R}$. So the process $\xi_t^{2\ell} + a_\ell(\tau-t)$; $0 \leq t \leq \tau$ is a nonnegative supermartingale and consequently, for any $\ell \geq 1$:

$$\sup_{0\leq t\leq\tau} E\xi_t^{2\ell} \leq x^{2\ell} + a_\ell\tau \tag{24}$$

$$P(T_n < \tau) = P(\sup_{0\leq t\leq\tau} |\xi_t| > n) \leq \frac{x^2 + a_\ell\tau}{n^{2\ell}}. \tag{25}$$

Therefore, $T_n \to \tau$, a.s. as $n \to \infty$, and

$$W(x,\tau) \geq EW(\xi_{T_n}, \tau-T_n) \xrightarrow[n\to\infty]{} EW(\xi_\tau, 0) = Ev(\xi_\tau) \geq 0,$$

because the family of random variables $W(\xi_t, \tau-t)$; $0 \leq t \leq \tau\}$ is uniformly integrable: $\sup_{0\leq t\leq\tau} EW^2(\xi_t, \tau-t) \leq \text{Const}(\tau + \sup_{0\leq t\leq\tau} E\xi_t^{2k}) < \infty$ by (24), some $k \geq 1$. Q.E.D.

PROPOSITION 3: $\underline{V_x(|x|,\tau) \leq v_x(|x|);\ (x,\tau) \in \mathbb{R} \times [0,T],\ \text{any}\ \ T > 0.}$

Proof: It is readily verified from equations (3), (19) by differentiation that the function:

$$M(x,\tau) \triangleq v_x(x) - V_x(x,\tau)$$

satisfies the equation:

$$M_\tau = \frac{1}{2} M_{xx} - v_x M_x - V_{xx} M; \quad (x,\tau) \in \mathbb{R} \times (0,T]$$

along with the initial condition:

$$M(x,0) = v_x(x); \quad x \in \mathbb{R}.$$

Note also that:

$$M(0,\tau) = 0; \quad 0 \leq \tau \leq T.$$

It follows by Itô's rule that

$$d\left[M(\xi_t,\tau-t)\cdot\exp\left\{-\int_0^t V_{xx}(\xi_s,\tau-s)ds\right\}\right] = \exp\left\{-\int_0^t V_{xx}(\xi_s,\tau-s)ds\right\} M_x(\xi_t,\tau-t)dw_t. \quad (26)$$

Take $x > 0$; the result follows by symmetry for $x < 0$ while it is obvious for $x = 0$. By the same argument as in Lemma 3, the integral of the right hand side of (26) over $[0,t]$ is a square-integrable martingale on $0 \leq t \leq \tau$, so that

$$v_x(x) - V_x(x,\tau) = E\left[v_x(\xi_R)\cdot\exp\left\{-\int_0^R V_{xx}(\xi_t,\tau-t)dt\right\}\right] \tag{27}$$

where

$$R = \inf\{0 \le t \le \tau;\ \xi_t = 0\}$$

$$= \tau, \text{ if } \ \xi_t > 0 \quad \text{for all} \quad 0 \le t \le \tau.$$

The result follows from (27).

COROLLARY 1: $P(|\xi_t| \le |\eta_t^\tau| \le |x+w_t|;\ \ 0 \le t \le \tau) = 1.$

Follows from the Ikeda-Watanable comparison Theorem [10], along with equations (15) and (21).

COROLLARY 2: $\lim_{\tau\to\infty} V_x(x,\tau) = v_x(x)$, uniformly on bounded x-sets.

Proof: From (27) we get for $x > 0$:

$$0 \le v_x(x) - V_x(x,\tau) \le Ev_x(\xi_R) = \int_{\{R=\tau\}} v_x(\xi_\tau)dP.$$

Due to Corollary 1:

$$P(R = \tau) \le P(x+w_t > 0, \text{ all } \ 0 \le t < \tau) = 2\Phi(x\tau^{-1/2}) - 1 \to 0 \quad \text{as} \quad \tau \to \infty,$$

uniformly on bounded x-sets ($\Phi(\cdot)$ is the cummulative normal distribution function). On the other hand, we get from (11), (16) that

$$\lim_{\tau\to\infty} Ev_x^2(\xi_t) = \int_{-\infty}^{\infty} v_x^2(y)(z^*(y))^2 dy \leq 2\lambda^*,$$

therefore $\sup_{\tau>0} Ev_x^2(\xi_\tau) \leq c < \infty$, which proves uniform integrability of the family $\{v_x(\xi_\tau);\ \tau > 0\}$, hence uniform absolute continuity, hence the result.

REMARK: Note that

$$\begin{aligned} d|\xi_t| &= -v_x(|\xi_t|)dt + d\tilde{w}_t + d\ell_t, \\ |\xi_0| &= |x| \end{aligned} \tag{28}$$

where ℓ_t is an increasing process that off $\{t,\xi_t = 0\}$ (the local time spent by the $\xi_.$ process at zero), and $\tilde{w}_t = \int_0^t \operatorname{sgn} \xi_s dw_s$ is a new Brownian Motion on $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$.

For a discussion of equation of the form (28) see Gihman and Skorohod [7, §23] as well as McKean [13;§§3.8, 3.9]. Application of the generalized Itô formula (Meyer [14, p. 365]) to $M(|\xi_t|,\tau-t)$ yields

$$\begin{aligned} dM(|\xi_t|,\tau-t) &= V_{xx}(|\xi_t|,\tau-t)M(|\xi_t|,\tau-t)dt + M_x(|\xi_t|,\tau-t)d\ell_t + M_x(|\xi_t|,\tau-t)d\tilde{w}_t \\ &= V_{xx}(|\xi_t|,\tau-t)M(|\xi_t|,\tau-t)dt + M_x(|\xi_t|,\tau-t)d\tilde{w}_t, \end{aligned}$$

because $M_x(|\xi_\tau|,\tau-t) = 0$ on $\{t;\xi_t = 0\}$, i.e. where $d\ell_t \neq 0$. A new application of Itô's rule gives (26) with ξ_t replaced by $|\xi_t|$, so finally if

$$R_t = \inf\{t \le s \le \tau;\ \xi_s = 0\}$$

$$= \tau, \text{ if } \xi_s \ne 0 \text{ for all } t \le s \le \tau,$$

we get

$$v_x(|\xi_t|) - V_x(|\xi_t|,\tau-t) = E\left[v_x(|\xi_{R_t}|)\cdot\exp\left\{-\int_t^{R_t} V_{xx}(|\xi_s|,\tau-s)ds\right\}\middle|\mathcal{F}_t\right]. \qquad (27)'$$

THEOREM 1: $\lim\limits_{\tau\to\infty} \frac{V(x,\tau)}{\tau} = \lambda^*$, uniformly on bounded x-sets.

Proof: By Corollary 1 and convexity of ϕ, V in x:

$$V(x,\tau) = E\int_0^\tau \{\phi(\eta_t^\tau) + \tfrac{1}{2}V_x^2(\eta_t^\tau,\tau-t)\} \ge E\int_0^\tau \{\phi(\xi_t) + \tfrac{1}{2}V_x^2(\xi_t,\tau-t)\}dt.$$

Therefore,

$$\begin{aligned}\frac{1}{\tau}E\int_0^\tau\{\phi(\xi_t) + \tfrac{1}{2}v_x^2(\xi_t)\,dt - \frac{1}{2\tau}\int_0^\tau E\{v_x^2(\xi_t) - V_x^2(\xi_t,\tau-t)\}dt \\ \le \frac{V(x,\tau)}{\tau} \le \lambda^* + \frac{v(x)}{\tau}.\end{aligned} \qquad (29)$$

The first term on the left hand side of (29) converges to λ^* as $\tau \to \infty$, uniformly on bounded x-sets. For the integrand of the second term we have the estimate

$$\tfrac{1}{2}E[v_x^2(\xi_t) - V_x^2(\xi_t,\tau-t)] \le E[v_x(|\xi_t|)v_x(|\xi_{R_t}|)] = \int_{\{R_t=\tau\}} v_x(|\xi_t|)v_x(|\xi_\tau|)dP, \qquad (30)$$

where Proposition 3, relation (27)' and the convexity of $V(\cdot,\tau)$ have been taken into account. Splitting the integral in (30) over the events $\{T_n = \tau\}$ and $\{T_n < t\}$, T_n as in (23), we get the estimate

$$v_x^2(n)\cdot P(T_n = R_t = \tau) + \int_{\{T_n<\tau\}} v_x(|u_t|)v_x(|\xi_\tau|)dP.$$

Now

$$P(T_n = R_t = \tau) = P(|\xi_t| + \tilde{w}_s - \tilde{w}_t + \ell_s - \ell_t - \int_t^s v_x(|\xi_u|)du > 0,$$
$$|\xi_s| < n; \text{ all } t \le s \le \tau)$$

$$\le P(n + \tilde{w}_s - \tilde{w}_t > 0, \text{ all } t \le s \le \tau) = 2\Phi(n(\tau-t)^{-1/2}) - 1 \le 2n(2\pi(\tau-t))^{-1/2}.$$

Therefore,

$$\frac{1}{2\tau} E\int_0^\tau \{v_x^2(\xi_t) - V_x^2(\xi_t,\tau-t)\}dt \le \text{const.}\tau^{-1/2}n^{2m+1}$$
$$+ \int_{\{T_n<\tau\}}\left\{\frac{1}{\tau}\int_0^\tau v_x(|\xi_t|)dt\right\}v_x(|\xi_\tau|)dP. \tag{31}$$

For the choice $\tau = n^{4m+3}$ the first term on the right hand side of (31) tends to zero as $n \to \infty$, while (25) with $\ell = 2(m+1)$ gives $P(T_n < n^{4m+3}) \to 0$ as $n \to \infty$, uniformly on bounded x-sets. On the other hand, the family of random variables

$$\left\{\frac{1}{\tau}\int_0^\tau v_x(|\xi_t|)dt\cdot v_x(|\xi_\tau|);\ \tau > 0\right\}$$ is uniformly integrable;

indeed, we have by the Cauchy inequality:

$$\frac{1}{\tau^2} E\left[v_x(|\xi_\tau|)\int_0^\tau v_x(|\xi_t|)dt\right]^2 \le \frac{1}{\tau} E\int_0^\tau v_x^2(|\xi_t|)dt \cdot Ev_x^2(|\xi_\tau|) \le c^2 < \infty$$

where c is an upper bound on $\sup_{\tau>0} Ev_x^2(\xi_\tau)$ (Corollary 2).

By uniform absolute continuity the second term on the right hand side of (31) also converges to zero as $\tau = n^{4m+3} \to \infty$, uniformly on bounded x-sets. Therefore, both left and right hand sides of the double inequality (29) converge to λ^* as $\tau \to \infty$, uniformly on bounded x-sets; Q.E.D.

EXAMPLE: In the case $\varphi(x) = x^2$, we have $\lambda^* = 1/\sqrt{2}$,

$z(x) = \text{const. } \exp(-\frac{x^2}{\sqrt{2}})$,

$$v(x) = x^2/\sqrt{2}, \quad V(x,\tau) = (a(\tau)x^2 + \int_0^\tau a(s)ds)/\sqrt{2},$$

with $a(\tau) = \tan(\tau\sqrt{2})$. The results of the present section are readily verified.

THEOREM 2: <u>The principal eigenvalue λ^* of the eigenvalue problem (1) admits the limiting-stochastic representation (2), uniformly on bounded x-sets</u>.

<u>Proof</u>: The change of variables $m = \exp(-V)$ transforms equation (19) into the linear equation with potential term:

$$m_\tau = \frac{1}{2} m_{xx} - \phi(x)m; \quad (x,\tau) \in \mathbb{R} \times (0,T]$$

$$m(x,0) = 1 \qquad ; \; x \in \mathbb{R}.$$

It is checked that

$$\left\{ m(x+w_t, \tau-t)\cdot\exp\left[-\int_0^t \phi(x+w_s)ds\right]; \; 0 \leq t \leq \tau \right\}$$

is a square-integrable martingale, hence the so called Feynman-Kac formula:

$$m(x,\tau) = E\left[\exp\left\{-\int_0^\tau \phi(x+w_s)ds\right\}\right].$$

(2) follows from the latter and Theorem 1.

THEOREM 3: <u>The control law $u_t^*(x) = -v_x(x_t)$; $x \in C[0,\infty)$, is optimal in $\mathcal{U}$ for the stationary control problem.</u>

<u>Proof</u>: Apply Itô's rule to $V(x_t^u, \tau-t)$, any $u \in \mathcal{U}$ and take expectations to obtain by virtue of conditions (5), (6) the inequality

$$V(x,\tau) \leq E_x^u \int_0^\tau \{\phi(x_t^u) + \frac{1}{2} u_t^2(x^u)\}dt.$$

Dividing by τ and letting $\tau \to \infty$ we get

$$\lambda^* \leq \varliminf_{\tau\to\infty} \frac{1}{\tau} E \int_0^\tau \{\phi(x_t^u) + \frac{1}{2} u_t^2(x^u)\}dt = J(u,x), \text{ any } x \in \mathbb{R}.$$

Since $\lambda^* = J(u^*)$ the optimality of u^* follows.

Acknowledgement: The author is indebted to Professor W.H. Fleming for a number of helpful discussions and suggestions.

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